Math 5C Discussion Problems 1

Line Integrals

1. For each of the following, compute $\int_C f\,ds$.

   (a) $f(x,y) = 2x - y$, C parametrized by $\mathbf{r}(t) = (e^t + 1, e^t - 2)$, $0 \le t \le \ln 2$

   (b) $f(x,y) = xy$, C parametrized by $\mathbf{r}(t) = (2\cos t, 3\sin t, 5t)$, $0 \le t \le \pi/2$

   (c) $f(x,y) = 1$, C is the first-quadrant portion of the curve $x^{2/3} + y^{2/3} = 1$

   (d) $f(x,y,z) = y - z^2$, C is parametrized by $\mathbf{r}(t) = (t^2, \ln t, 2t)$, $1 \le t \le 4$

   (e) $f(x,y,z) = xy$, C is the straight-line path from $(0,0)$ to $(1,1)$ followed by the straight-line path from $(1,1)$ to $(2,3)$

   (f) $f(x,y,z) = (x+y+z)/(x^2+y^2+z^2)$, C is the straight-line path from $(1,1,1)$ to $(2,2,2)$

2. Let C be the circle of radius 4 centered at the origin in $\mathbb{R}^2$. Without integrating, evaluate

$$\int_C \exp(x^2 + y^2)\,ds.$$

3. For each of the following, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

   (a) $\mathbf{F} = (y^2, -x^2)$, C is the part of the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$

   (b) $\mathbf{F} = (2xy, e^y)$, C is parametrized by $\mathbf{r}(t) = 4t^3, t^2$, $0 \le t \le 1$

   (c) $\mathbf{F} = (xy, yz, xz)$, C is parametrized by $\mathbf{r}(t) = (\sin t \cos t, t^2)$, $0 \le t \le \pi/2$

4. Let C be an oriented smooth curve in $\mathbb{R}^3$. Let $\mathbf{T}$ be the unit tangent vector of C (in the direction of the orientation) and $\mathbf{N}$ be the unit normal vector of C. Evaluate

$$\int_C \mathbf{T} \cdot d\mathbf{r} \qquad \text{and} \qquad \int_C \mathbf{N} \cdot d\mathbf{r}.$$

5. Evaluate $\int_C \frac{-y\,dx + x\,dy}{x^2 + y^2}$, where C is the circle $x^2 + y^2 = 1$, oreinted counterclockwise.

6. Evaluate $\int_C xy\,dx + x^2\,dy$, where C is the striaght-line path from $(0,0)$ to $(1,1)$.

7. Rewrite $\int_C e^y\,dx + \sin(xy)\,dy$ in the form $\int_C \mathbf{F} \cdot d\mathbf{r}$.

8. Let C be the straight-line path from $(1,1,1)$ to $(1,2,4)$. Compute

$$\int_C 2xyz\,dx + x^2 z\,dy + x^2 y\,dz.$$

9. Let C be the series of straight-line paths from $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$. Compute

$$\int_C z\,dx + x\,dy + y\,dz.$$

Double Integrals

1. Evaluate the following integrals.

   (a) $\int_0^1 \int_0^x \cos(x^2)\, dy\, dx$

   (b) $\int_1^2 \int_0^{y/2} x\sqrt{x^2+y^2}\, dx\, dy$

2. For each of the following, evaluate $\iint_R f\, dA$.

   (a) $f(x,y) = xy$, $R = [0,1] \times [1,2]$

   (b) $f(x,y) = x^{-2/3}$, R is the region bounded by $y = x^2$ and $y = 4 - x^2$

   (c) $f(x,y) = \ln(xy)$, R is the trianglular region bounded by $x = 0$, $y = 1$, and $y = x$

3. Evaluate the following integrals. Consider interchanging the order of integration.

   (a) $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$

   (b) $\int_0^1 \int_{y^{1/3}}^1 e^{x^4}\, dx\, dy$

   (c) $\int_0^3 \int_{x^2}^9 x\cos(2y^2)\, dy\, dx$

4. Suppose that $\int_0^1 f(x)\, dx = A$ and $\int_0^1 g(y)\, dy = B$. What is $\iint_{[0,1]^2} f(x)g(y)\, dA$?

5. Let D be the unit disk in $\mathbb{R}^2$ centered at the origin. What is $\iint_D 4\, dA$?

6. Let R be the parallelogram in $\mathbb{R}^2$ with vertices $(0,0), (2,2), (1,0), (3,2)$. What is $\iint_D -2\, dA$?

7. For each of the following, evaluate the integral. Consider polar coordinates.

   (a) $\iint_D \frac{1}{\sqrt{x^2+y^2}}\, dA$, where D is the unit disk centered at the origin

   (b) $\iint_D (x^2+y^2)^{3/4}\, dA$, where D is the disk centered at the origin with radius 4

   (c) $\iint_D \frac{\sqrt{x^2+y^2}}{x^2}\, dA$, where D is the top half ($y \geq 0$) of the unit disk centered at the origin

   (d) $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\, dy\, dx$

   (e) $\iint_T \frac{1}{\sqrt{x^2+y^2}}\, dA$, where T is the triangle bounded by $y = 0$, $x = 1$, and $y = x$

   (f) $\iint_D (x^2+y^2)^{-1/2}\, dA$, where D is the unit disk centered at $(1,0)$

   (g) $\iint_R x\, dA$, where R is the annular region between $x^2+y^2 = 1$ and $x^2+y^2 = 4$

8. Consider the integral $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA$.

   (a) Evaluate the integral.

   (b) Use the result to evaluate $\int_0^\infty e^{-x^2}\, dx$.

Surfaces and Their Integrals

1. For the surface parametrized by

$$x = \cos v \sin u, \qquad y = \sin v \sin u, \qquad z = \cos u$$

with $0 \le u \le \pi$ and $0 \le v \le 2\pi$, compute an expression for the unit normal vector in terms of u and v as well as the surface area. Identify the surface.

2. Repeat the previous problem for the parametrization

$$x = \sin v, \qquad y = u, \qquad z = \cos v$$

with $0 \le v \le 2\pi$ and $-1 \le u \le 3$.

3. Let D be the unit disk (centered at the origin) in the u-v plane. Find the area of the surface parametrized by

$$x = u - v, \qquad y = u + v, \qquad z = uv,$$

where $(u, v) \in D$.

4. Find the area of the surface $z = (2/3)(x^{3/2} + y^{3/2})$ which lies above the unit square $[0, 1] \times [0, 1]$.

5. Let S be the surface with $x^2 + y^2 \le 1$ and $z = x^2 + y^2$.

   (a) Find the area of S.

   (b) Evaluate $\displaystyle\iint_S z \, d\sigma$.

   (c) Evaluate $\displaystyle\iint_S (x, y, z) \cdot d\mathbf{A}$.

6. Let S be the sphere of radius R centered at the origin. Without computation:

   (a) Evaluate $\displaystyle\iint_S d\sigma$.

   (b) Evaluate $\displaystyle\iint_S (x^2 + y^2 + z^2) \, d\sigma$.

   (c) Evaluate $\displaystyle\iint_S (x^2 + y^2 - 2z^2) \, d\sigma$.

   (d) Evaluate $\displaystyle\iint_S x^2 \, d\sigma$.

7. Let S be the cylinder $x^2 + z^2 = 2$, $0 \le y \le 2$. Given $f(x, y, z) = x^2 + z^2$, compute the flux of ∇f out of S.

8. Let R be the solid region $x^2 + y^2 \le 1$, $0 \le z \le 1$. Compute the flux of $(1, 1, z(x^2 + y^2)^2)$ out of R.

9. Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ with $1 \le z \le 2$ and downward-pointing normal vector. Compute

$$\iint_S (x^2, y^2, z^2) \cdot d\mathbf{A}.$$

10. Let S be the surface in $\mathbb{R}^3$ given by $x^2 + y^2 = 1$, $0 \le z \le 1$, and $x \ge 0$. Assume that S is oriented with normal in the positive x direction. Find the flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i}$ across S.

11. Let S be the unit disk $x^2 + y^2 \le 1$, $z = 0$, upwardly oriented.

    (a) Without computing, what is the unit normal to S?

    (b) Evaluate $\displaystyle\iint_S (\arctan(xy), \sqrt{x^2 + y^2}, xyz) \cdot d\mathbf{A}$.

Triple Integrals

1. Let R be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $z = x + y$.
   (a) Find the volume of R.
   (b) Evaluate $\iiint_R x\,dV$.
   (c) Evaluate $\iiint_R z\,dV$.
2. Rewrite $\int_0^1 \int_0^x \int_0^y f(x, y, z)\,dz\,dy\,dx$ in the order $dz\,dy\,dx$. Sketch the region.
3. Let R be the pyramid with base vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)$ and apex $(0, 0, 1)$.
   (a) Evaluate $\iiint_R (1 - z^2)\,dV$.
   (b) Evaluate $\iiint_R (x^2 + y^2)\,dV$.
4. Integrate $\sqrt{x^2 + y^2 + z^2} e^{-(x^2+y^2+z^2)}$ over the unit ball centered at the origin.
5. Let R be the region $1 \le x^2 + y^2 + z^2 \le 4$.
   (a) Evaluate $\iiint_R \frac{dV}{\sqrt{x^2 + y^2 + z^2}}$.
   (b) Evaluate $\iiint_R xyz\,dV$.
6. Evaluate $\iiint_{\mathbb{R}^3} \frac{dV}{1 + (x^2 + y^2 + z^2)^3}$.
7. Evaluate $\iiint_{\mathbb{R}^3} e^{-z^2} \frac{\sqrt{x^2 + y^2}}{1 + (x^2 + y^2)^{3/2}}\,dV$.
8. A circular hole of radius 1 is drilled throught the center of a sphere of radius 2. How much volume remains?
9. Find the volume of the 'ice cream cone' defined by $\sqrt{x^2 + y^2} \le z$ and $x^2 + y^2 + z^2 \le 1$.
10. Find the volume of the 'ice cream cone' defined by $\sqrt{x^2 + y^2} \le z$ and $x^2 + y^2 + z^2 \le z$.
11. Find the volume of the region $x^2 + y^2 \le 1$, $0 \le z \le \sqrt{x^2 + y^2}$.
12. Integrate x^2 over the region $0 \le z \le 1 - \sqrt{x^2 + y^2}$.
13. Let B be the ball of radius R centered at the origin. Without computation, evaluate the following integrals.
   (a) $\iiint_B 3\,dV$
   (b) $\iiint_B x\,dV$
   (c) $\iiint_B (1 - x^2)\,dV$
   (d) $\iiint_B (y - z)\,dV$
   (e) $\iiint_B xyz\,dV$